# Some properties for superprocess under a stochastic flow

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**Abstract**

For a superprocess under a stochastic flow, we prove that it has a density with respect to the Lebesgue measure for $d = 1$ and is singular for $d > 1$. For $d = 1$, a stochastic partial differential equation is derived for the density. The regularity of the solution is then proved by using Krylov's $L_p$-theory for linear SPDE. A snake representation for this superprocess is established. As applications of this representation, we prove the compact support property for general $d$ and singularity of the process when $d > 1$.

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# 1 Introduction

Superprocesses under stochastic flows have been studied by many authors since the work of Wang ([11],[12]) and Skoulakis and Adler [9]. At an early stage, this problem was studied as the high-density limit of a branching particle system while the motion of

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each particle is governed by an independent Brownian motion as well as by a common Brownian motion which determines the stochastic flow. The limit is characterized by a martingale problem whose uniqueness is established by a moment duality. Before we go any further, let us introduce the model in more detail.

Let $b: \mathbb{R}^d \to \mathbb{R}^d$, $\sigma_1, \sigma_2: \mathbb{R}^d \to \mathbb{R}^{d\times d}$ be measurable functions. Let $W, B_1, B_2, \cdots$ be independent $d$-dimensional Brownian motions. Consider a branching particle system performing independent binary branching. Between branching times, the motion of the $i$th particle is governed by the following stochastic differential equation (SDE):

$$d\eta_i(t) = b(\eta_i(t))dt + \sigma_1(\eta_i(t))dW(t) + \sigma_2(\eta_i(t))dB_i(t). \tag{1.1}$$

It is proved by Skoulakis and Adler [9] that the high-density limit $X_t$ is the unique solution to the following martingale problem (MP): $X_0 = \mu \in \mathcal{M}_F(\mathbb{R}^d)$, where $\mathcal{M}_F(\mathbb{R}^d)$ denotes the space of finite nonnegative measures on $\mathbb{R}^d$ and for any $\phi \in C_0^2(\mathbb{R}^d)$,

$$M_t(\phi) \equiv \langle X_t, \phi\rangle - \langle \mu, \phi\rangle - \int_0^t \langle X_s, L\phi\rangle\, ds \tag{1.2}$$

is a continuous martingale with quadratic variation process

$$\langle M(\phi)\rangle_t = \int_0^t \left(\left\langle X_s, \phi^2\right\rangle + \left|\left\langle X_s, \sigma_1^T \nabla\phi\right\rangle\right|^2\right) ds \tag{1.3}$$

where

$$L\phi = \sum_{i=1}^d b^i \partial_i \phi + \frac{1}{2}\sum_{i,j=1}^d a^{ij}\partial_{ij}^2\phi,$$

$a^{ij} = \sum_{k=1}^d \sum_{\ell=1}^2 \sigma_\ell^{ik}\sigma_\ell^{kj}$, $\partial_i$ means the partial derivative with respect to the $i$th component of $x \in \mathbb{R}^d$, $\sigma_1^T$ is the transpose of the matrix $\sigma_1$, $\nabla = (\partial_1, \cdots, \partial_d)^T$ is the gradient operator and $\langle \mu, f\rangle$ represents the integral of the function $f$ with respect to the measure $\mu$. It was conjectured in [9] that the conditional log-Laplace transform of $X_t$ should be the unique solution to a nonlinear stochastic partial differential equation (SPDE). Namely

$$\mathbb{E}_\mu\left(e^{-\langle X_t, f\rangle}\middle| W\right) = e^{-\langle \mu, y_{0,t}\rangle} \tag{1.4}$$

and

$$\begin{aligned} y_{s,t}(x) &= f(x) + \int_s^t \left( L y_{r,t}(x) - y_{r,t}(x)^2 \right) dr \\ &\quad + \int_s^t \nabla^T y_{r,t}(x) \sigma_1(x) \hat{d} W(r) \end{aligned} \tag{1.5}$$

where $\hat{d}W(r)$ represents the backward Itô integral:

$$\int_s^t g(r)\hat{d}W(r) = \lim_{|\Delta|\to 0} \sum_{i=1}^n g(r_i)\left(W(r_i) - W(r_{i-1})\right)$$

where $\Delta = \{r_0, r_1, \cdots, r_n\}$ is a partition of $[s,t]$ and $|\Delta|$ is the maximum length of the subintervals.

This conjecture was confirmed by Xiong [13] under the following conditions (BC) which will be assumed throughout this paper: $f \geq 0$, $b$, $\sigma_1$, $\sigma_2$ *are bounded with bounded first and second derivatives.* $\sigma_2^T\sigma_2$ *is uniformly positive definite,* $\sigma_1$ *has third continuous bounded derivatives.* $f$ *is of compact support.*

Making use of the conditional log-Laplace functional, the long-term behavior of this process is studied in [14]. Also, the model has been extended in that paper to allow infinite measures $\mu \in \mathcal{M}_{tem}(\mathbb{R}^d)$, namely, $\int_{\mathbb{R}^d} e^{-\lambda|x|}\mu(dx) < \infty$ for some $\lambda > 0$. We shall assume $\mu \in \mathcal{M}_{tem}(\mathbb{R}^d)$ throughout this paper. A similar model has been investigated by Wang [12] and Dawson et al [1] when the spatial dimension is 1. Further, in that case, it is proved by Dawson et al [2] that their process is density-valued and solves a SPDE. The regularity of the solution was left *open* in that article.

This paper is organized as follows: In Section 2, we establish a snake representation for $X_t$. As immediate consequences to this representation, we get the compact support property of $X_t$ (for all $d$) and for $d > 1$, $X_t$ takes values in the set of singular measures. Then, for $d = 1$, we prove in Section 3 that $X_t$ is absolutely continuous with respect to Lebesgue measure and show that the density $X(t,x)$ satisfies the following SPDE

$$\partial_t X = L^* X - \partial_x(\sigma_1 X)\dot{W}_t + \sqrt{X}\dot{B}_{tx} \tag{1.6}$$

where $B$ is a Brownian sheet and $L^*$ is the adjoint operator of $L$. The main result of this paper is to show the Hölder continuity of $X(t,x)$.

Here is the main result. First recall that for $n \in \mathbb{R}$ and $p \in [2,\infty)$, $H_p^n$ is the space of Bessel potentials with norm

$$\|u\|_{n,p} = \|(I-\Delta)^{n/2}u\|_p.$$

**Theorem 1.1** *Suppose that Condition (BC) is satisfied. Then*

*i) If $d>1$, then $X_t$ is singular a.s.*

*ii) If $d=1$, then $X_t$ is absolutely continuous with respect to Lebesgue measure and the density satisfies the SPDE (1.6).*

*iii) If in addition, $\mu$ satisfies $\mu \in H_p^{\frac{1}{2}-\epsilon-2/p}$ with $\epsilon \in (0,\frac{1}{4})$ and $p > \frac{1}{\epsilon}$ and also satisfies*

$$\sup_{t,x} \langle \mu, \varphi_t(x-\cdot)\rangle < \infty, \tag{1.7}$$

*then the density $X(t,x)$ is Hölder continuous in $x$ with index $\frac{1}{2}-2\epsilon$ for (a.e.) $t$ a.s., where $\varphi_t(x)$ is the density of a normal random variable with mean 0 and variance $t$.*

Note that (1.7) is satisfied if $\mu$ has bounded density with respect to Lebesgue measure.

Suppose that we apply the usual integral equation as in [10], Chapter 3, for (1.6) in order to prove the Hölder continuity. Then formally we have

$$\begin{aligned} X(t,x) &= \int p_0(t,x,y)X(0,y)dy + \int_0^t \int \sigma_1(y)X(s,y)\partial_y p_0(t-s,x,y)dydW(s) \\ &\quad + \int_0^t \int \sqrt{X(s,y)}p_0(t-s,x,y)B(dsdy) \end{aligned}$$

where $p_0$ is the transition function of the Markov process with generator $L$. However, the second term on the right hand side of the above equation is about

$$\int_0^t (t-s)^{-1/2}dW(s)$$

which is *not* convergent. Therefore, the convolution argument used by Konno and Shiga [5] does not apply to our model. In Section 4, we freeze the nonlinear term in (1.6) and apply Krylov's $L_p$-theory for linear SPDE to get the Hölder continuity with index slightly less than $\frac{1}{2}$ for $X$.

Note that the SPDE in [2] is (1.6) in current paper with $\dot{W}_t$ replaced by a space-time noise which is colored in space and white in time. The method of this paper can be applied to that equation to prove the regularity for its solution.

# 2 Snake representation

In this section, we construct a path-valued process $\mathcal{Y}_t$ such that the process $X_t$ can be represented according to this process. Then, as an easy application of this representation, we derive the properties for $X_t$.

For the convenience of the reader, we recall some basic definitions and facts taken from Le Gall [8]. Let $\zeta \geq 0$ and let $f$ be a continuous function from $\mathbb{R}_+$ to $\mathbb{R}^d$ such that $f(s) = f(\zeta)$, $\forall\ s \geq \zeta$. We call such pair $(f, \zeta)$ a stopped path with $\zeta$ being the lifetime of the path. We denote the collection of all stopped paths by $\mathbb{W}$. For $(f, \zeta)$, $(f', \zeta') \in \mathbb{W}$, define a distance

$$\delta((f,\zeta),(f',\zeta')) = \sup_{s\geq 0} |f(s) - f'(s)| + |\zeta - \zeta'|.$$

Then $(\mathbb{W}, \delta)$ is a Polish space. In [8], Le Gall constructed a continuous time-homogeneous strong Markov process $(\mathcal{Z}_t, \zeta_t)$ taking values on $\mathbb{W}$. $\zeta_t$ is a one-dimensional reflecting Brownian motion. Given $\zeta$, the process $\mathcal{Z}$ has the following property: for all $r < t$, and for all $s \leq m_{r,t} := \inf_{r\leq u\leq t} \zeta_u$ we have $\mathcal{Z}_r(s) = \mathcal{Z}_t(s)$. Furthermore, given $m_{r,t}$ and $\mathcal{Z}_r(m_{r,t})$, the processes $\mathcal{Z}_r(s) : s \geq m_{r,t}$ and $\mathcal{Z}_t(s) : s \geq m_{r,t}$ are conditionally independent Brownian motions with lifetimes $\zeta_r$ and $\zeta_t$ respectively.

Denote the strong solution to the SDE

$$d\eta(t) = b(\eta(t))dt + \sigma_1(\eta(t))dW(t) + \sigma_2(\eta(t))dB(t)$$

by $\eta(t) = F(t, W, B)$. Define the following path-valued process

$$\mathcal{Y}_t(s) = F(s, W, \mathcal{Z}_t)$$

with the life-time process $\zeta_t$.

**Lemma 2.1** $(\mathcal{Y}_t, \zeta_t)$ *is a continuous* $\mathbb{W}$*-valued process.*

Proof: Note that for all $r < t$ and for all $s < m_{r,t}$, we have $\mathcal{Y}_r(s) = \mathcal{Y}_t(s)$. Furthermore, for given $\mathcal{Y}_r(m_{r,t})$, the processes $\mathcal{Y}_r(s) : s \geq m_{r,t}$ and $\mathcal{Y}_t(s) : s \geq m_{r,t}$ are the motions of two particles (say, $\eta_1$ and $\eta_2$) given as in the introduction with lifetimes $\zeta_r$ and $\zeta_t$ starting from the same position $\mathcal{Y}_r(m_{r,t})$. A simple application of Burkholder's inequality gives

$$\mathbb{E}\left[\sup_{m\leq s\leq M} |\eta_1(s) - \eta_2(s)|^k\right] \leq K|M - m|^{k/2},$$

where $m = m_{r,t}$ and $M = \zeta_r \vee \zeta_t$. Denote by $\mathbb{E}^{\,\zeta}$ the conditional expectation given $\zeta$. Then

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\geq 0} |\mathcal{Y}_r(s) - \mathcal{Y}_t(s)|^k\right] &= \mathbb{E}\left[\mathbb{E}^{\,\zeta}\left\{\sup_{s\geq m_{r,t}} |\mathcal{Y}_r(s) - \mathcal{Y}_t(s)|^k\right\}\right] \\
&\leq \mathbb{E}\left[K|\zeta_r + \zeta_t - 2m_{r,t}|^{k/2}\right] \\
&\leq K\mathbb{E}\left[\sup_{s\in[r,t]} |\zeta_s - \zeta_r|^{k/2}\right] \\
&\leq K|t - r|^{k/4}.
\end{aligned}$$

The conclusion follows from Kolmogorov's criteria by taking $k > 4$; see [10] for Kolmogorov's criteria. ∎

**Theorem 2.2**

$$X_t(f)=\int_t^\tau f(\mathcal{Y}_s(\zeta_s))d\ell_s^t \tag{2.1}$$

*where $\ell^t$ is the local time process of $\zeta$ at level $t$ and*

$$\tau=\inf\{s:\ \ell_s^0\geq 1\}.$$

Proof: Fix a parameter $h>0$. For every $t\geq 0$, denote by $[a_t^1,b_t^1]$, $[a_t^2,b_t^2]$, $\cdots$, $[a_t^{N_t},b_t^{N_t}]$ the excursion intervals of $(\zeta_s)_{0\leq s\leq\tau}$ above level $t$, corresponding to excursions of height greater than $h$. Set

$$X_t^h=2h\sum_{i=1}^{N_t}\delta_{\mathcal{Y}_{a_t^i}(t)}.$$

Then $X_t^h$ is the measure-valued process corresponding to the branching particle system described as follows: At time $t=0$, we have $N_0$ particles in $\mathbb{R}^d$ with Poisson random measure with intensity measure $h^{-1}\mu$. The particles then move according to (1.1) with common $W$ and independent $B_i$'s. Each of them has a finite lifetime (independent of others) which is exponential with mean $h$. When a particle dies, it gives rise to either 0 or 2 new particles with probability $\frac{1}{2}$. The new particles start from the position of the their father. As in the proof of Theorem 2.1 in [8], by the well-known approximation of Brownian local time by upcrossing numbers, we have that $X_t^h$ converges weakly to $X_t$, where $X_t$ is given by the right hand side of (2.1). ∎

As an application of the snake representation, we have the following immediate consequence.

**Corollary 2.3** *If $\mu$ is a finite measure, then for any $t>0$, $X_t$ has compact support a.s.*

Proof: By the snake representation, there exists a finite set $I$ such that

$$\langle X_t,f\rangle=\sum_{i\in I}\int_0^{\tau_i} f(\hat{\mathcal{Y}}_s^i)d\ell_s^t(\zeta^i)$$

where $\hat{\mathcal{Y}}_s^i$ is the tip of the $i$th snake. It is not hard to show that $\hat{\mathcal{Y}}_s^i$ is continuous and hence, for any $t_0 > 0$,

$$\bigcup_{t\geq t_0} \mathrm{supp}(X_t) \subset \overline{\bigcup_{i\in I} \mathrm{Range}\left(\hat{\mathcal{Y}}^i\right)} = \bigcup_{i\in I} \{\hat{\mathcal{Y}}_s^i : \ 0 \leq s \leq \tau_i\} \tag{2.2}$$

is compact. ∎

To consider the case for $\mu$ being $\sigma$-finite, the following conditional martingale problem (CMP) is useful. The following lemma was proved in [14].

**Lemma 2.4** *i) If $X_t$ is the solution to MP, then there exists a Brownian motion $W_t$ such that for any $\phi \in C_0^2(\mathbb{R}^d)$,*

$$N_t(\phi) \equiv \langle X_t, \phi\rangle - \langle \mu, \phi\rangle - \int_0^t \langle X_s, L\phi\rangle \, ds - \int_0^t \left\langle X_s, \sigma_1^T \nabla\phi\right\rangle dW_s \tag{2.3}$$

*is a continuous $(\mathbb{P}, \mathcal{G}_t)$-martingale with quadratic variation process*

$$\langle N(\phi)\rangle_t = \int_0^t \left\langle X_s, \phi^2\right\rangle ds \tag{2.4}$$

*where $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{F}_\infty^W$.*

*ii) If $X_t$ is a solution to CMP, then it is a solution to MP.*

As another application of the snake representation, we have

**Corollary 2.5** *If $d \geq 2$, then $X_t$ is singular.*

Proof: If $\mu$ is finite and $d > 1$, it follows from (2.2) the support is of Lebesgue measure 0 since $\{\hat{\mathcal{Y}}_s^i : \ 0 \leq s \leq \tau_i\}$ is a continuous (one-dimensional) curve in $\mathbb{R}^d$. If $\mu$ is $\sigma$-finite, we can take $\mu = \sum_{n=1}^\infty \mu^n$ with $\mu^n$ finite. Construct the solution $X_t^n$ to CMP with the same $W$ and with initial $\mu^n$, $n = 1, 2, \cdots$. Then

$$X_t = \sum_{n=1}^\infty X_t^n$$

is the solution to CMP with initial $\mu$. Then $\mathrm{supp}(X_t^n)$ has Lebesgue measure 0 and hence, so does the support of $X_t$. This implies that $X_t$ is a singular measure a.s. ∎

# 3 SPDE for $d = 1$

In this section, we prove that $X_t$ has a density which satisfies the SPDE (1.6) whose mild form is

$$\begin{aligned}\langle X_t, f\rangle &= \langle \mu, f\rangle + \int_0^t \langle X_s, Lf\rangle\, ds + \int_0^t \langle X_s, \sigma_1 f'\rangle\, dW_r \\ &\quad + \int_0^t \int_{\mathbb{R}} \sqrt{X_s(x)} f(x) B(dsdx). \end{aligned} \tag{3.1}$$

Let $p_0(t,x,y)$ and $q_0(t,(x_1,x_2),(y_1,y_2))$ be the transition density functions of the Markov processes $\eta_1(t)$ and $(\eta_1(t),\eta_2(t))$ respectively. By Theorem 1.5 of [13], we have

$$\mathbb{E}\Big[\langle X_t, f\rangle\Big] = \int_{\mathbb{R}^2} f(y) p_0(t,x,y) dy \mu(dx) \tag{3.2}$$

and

$$\begin{aligned}&\mathbb{E}\Big[\langle X_t, f\rangle \langle X_t, g\rangle\Big] \\ &= \int_{\mathbb{R}^4} f(y_1) g(y_2) q_0(t,(x_1,x_2),(y_1,y_2)) dy_1 dy_2 \mu(dx_1)\mu(dx_2) \\ &\quad +2\int_0^t ds \int_{\mathbb{R}^4} p_0(t-s,z,y) f(z_1) g(z_2) q_0(s,(y,y),(z_1,z_2)) dz_1 dz_2 dy \mu(dz).\end{aligned} \tag{3.3}$$

**Theorem 3.1** *If $\mu(\mathbb{R}) < \infty$, then $X_t \in H_0 \equiv L^2(\mathbb{R})$ a.s.*

Proof: Take $f = p_0(\epsilon, x, \cdot)$ and $g = p_0(\epsilon', x, \cdot)$ in (3.3). Note that as $\epsilon, \epsilon' \to 0$,

$$\begin{aligned}&\int_{\mathbb{R}^2} p_0(\epsilon,x,z_1) p_0(\epsilon',x,z_2) p_0(t-s,z,y) q_0(t,(y,y),(z_1,z_2)) dz_1 dz_2 \\ \to\ & p_0(t-s,z,y) q_0(t,(y,y),(x,x)).\end{aligned}$$

Note that by Theorem 6.4.5 in Friedman [3], we have

$$p_0(\epsilon,x,y) \le c\varphi_{c'\epsilon}(x-y),$$

$$q_0(s,(y,y),(z_1,z_2)) \le c\varphi_{c's}(y-z_1)\varphi_{c's}(y-z_2)$$

where $\varphi_t(x)$ is the normal density with mean 0 and variance $t$ (introduced earlier). Note that $c'$ is a constant which is usually greater than 1. Since it does not play an essential role, to simplify the notations, we assume $c' = 1$ throughput the rest of this paper. Hence,

$$\begin{aligned}
&\int_{\mathbb{R}^2} p_0(\epsilon, x, z_1) p_0(\epsilon', x, z_2) p_0(t-s, z, y) q_0(s, (y, y), (z_1, z_2)) dz_1 dz_2 \\
&\leq \; c \int_{\mathbb{R}^2} \varphi_\epsilon(x - z_1) \varphi_{\epsilon'}(x - z_2) \varphi_{t-s}(z - y) \varphi_s(y - z_1) \varphi_s(y - z_2) dz_1 dz_2 \\
&= \; c \varphi_{s+\epsilon}(x - y) \varphi_{s+\epsilon'}(x - y) \varphi_{t-s}(z - y).
\end{aligned}$$

As

$$\begin{aligned}
&\lim_{\epsilon, \epsilon' \to 0} \int_0^T dt \int dx \int_0^t ds \int_{\mathbb{R}^2} \varphi_{s+\epsilon}(x - y) \varphi_{s+\epsilon'}(x - y) \varphi_{t-s}(z - y) dy \mu(dz) \\
&= \; \lim_{\epsilon, \epsilon' \to 0} \int_0^T dt \int_0^t ds \varphi_{2s+\epsilon+\epsilon'}(0) \mu(\mathbb{R}) \\
&= \; \int_0^T dt \int_0^t ds \varphi_{2s}(0) \mu(\mathbb{R}) \\
&= \; \int_0^T dt \int dx \int_0^t ds \int_{\mathbb{R}^2} \varphi_{t-s}(z - y) \varphi_s(x - y) \varphi_s(x - y) dy \mu(dz),
\end{aligned}$$

by the dominated convergence theorem, we see that as $\epsilon, \epsilon' \to 0$,

$$\begin{aligned}
&\int_0^T dt \int dx \int_0^t ds \int_{\mathbb{R}^4} p_0(t-s, z, y) p_0(\epsilon, x, z_1) p_0(\epsilon', x, z_2) q_0(s, (y, y), (z_1, z_2)) dz_1 dz_2 dy \mu(dz) \\
\to \; &\int_0^T dt \int dx \int_0^t ds \int_{\mathbb{R}^2} p_0(t-s, z, y) q_0(t, (y, y), (x, x)) dy \mu(dz).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\int_0^T dt \int dx \int_{\mathbb{R}^4} p_0(\epsilon, x, y_1) p_0(\epsilon', x, y_2) q_0(t, (x_1, x_2), (y_1, y_2)) dy_1 dy_2 \mu(dx_1) \mu(dx_2) \\
\to \; &\int_0^T dt \int dx \int_{\mathbb{R}^2} q_0(t, (x_1, x_2), (x, x)) \mu(dx_1) \mu(dx_2).
\end{aligned}$$

Hence

$$\int_0^T dt \int dx \mathbb{E} \; (\langle X_t, p(\epsilon, x, \cdot) \rangle \langle X_t, p(\epsilon', x, \cdot) \rangle)$$

$$
\begin{aligned}
\to & \int_0^T dt \int dx \int_{\mathbb{R}^2} q_0(t,(x_1,x_2),(x,x))\mu(dx_1)\mu(dx_2) \\
& + \int_0^T dt \int dx \int_0^t ds \int_{\mathbb{R}^2} p_0(t-s,x,y) q_0(t,(y,y),(x,x)) dy \mu(dx).
\end{aligned}
$$

From this, we can show that $\{\langle X_t, p_0(\epsilon, x, \cdot)\rangle : \epsilon > 0\}$ is a Cauchy sequence in $L^2(\Omega \times [0,T] \times \mathbb{R})$. This implies the existence of the density $X_t(x)$ of $X_t$ in $L^2(\Omega \times [0,T] \times \mathbb{R})$.

■

Next theorem considers infinite measure.

**Theorem 3.2** *If $\mu \in \mathcal{M}_{tem}(\mathbb{R}^d)$, then $X_t$ has a density $X_t(x)$.*

Proof: If $\mu$ is $\sigma$-finite, we can construct $X^n$ with $X_0^n = \mu^n$ being finite as those in the proof of Corollary 2.5. Then

$$
X_t = \sum_{n=1}^{\infty} X_t^n
$$

is the solution to CMP with initial $\mu$. Let

$$
X_t(x) = \sum_{n=1}^{\infty} X_t^n(x). \tag{3.4}
$$

By (3.2), we have

$$
\mathbb{E}\ X_t^n(x) = \int_{\mathbb{R}} p_0(t,y,x)\mu^n(dy).
$$

As

$$
p_0(t,x,y) \le c\varphi_t(x-y) \le c(t,\lambda,x)e^{-\lambda|y|},
$$

for any $\lambda > 0$, we have

$$
\begin{aligned}
\mathbb{E} \sum_{n=1}^{\infty} X_t^n(x) &= \sum_{n=1}^{\infty} \int_{\mathbb{R}} p_0(t,y,x)\mu^n(dy) \\
&= \int_{\mathbb{R}} p_0(t,y,x)\mu(dy) < \infty.
\end{aligned}
$$

Hence, $X_t(x)$ is well-defined by (3.4). It is then easy to show that $X_t(x)dx = X_t(dx)$.

■

Finally, we derive the SPDE satisfied by the density.

**Theorem 3.3** *If $d = 1$, then $X_t$ is the (weak) unique solution to the SPDE (3.1).*

Proof: Note that $N_t(\phi)$ is a continuous $(\mathbb{P}, \mathcal{G}_t)$-martingale with quadratic variation process

$$\langle N(\phi)\rangle_t = \int_0^t \int_{\mathbb{R}} \left(\sqrt{X_s(x)}\phi(x)\right)^2 dxds.$$

By the martingale representation theorem ([4], Theorem 3.3.5), there exists an $L^2(\mathbb{R})$-cylindrical Brownian motion $\tilde{B}$ on an extension of $(\Omega, \mathcal{F}, \mathcal{G}_t, \mathbb{P})$ such that

$$N_t = \int_0^t \left\langle \sqrt{X_s}, d\tilde{B}_s \right\rangle_{L^2(\mathbb{R})}.$$

There exists a standard Brownian sheet $B$ such that

$$\tilde{B}_t(h) = \int_0^t \int_{\mathbb{R}} h(x)B(dsdx), \qquad \forall h \in L^2(\mathbb{R}).$$

Therefore,

$$N_t(\phi) = \int_0^t \int_{\mathbb{R}} \sqrt{X_s(x)}\phi(x)B(dsdx).$$

As $B$ is a Brownian sheet on an extension of $\mathcal{G}_t$, it is easy to show that $B$ is independent of $W$. ∎

# 4 Hölder Continuity

This section is devoted to the proof of the main result: Theorem 1.1 (iii). Namely, in this section, we consider the regularity of the solution to the nonlinear SPDE (1.6). We use the linearization and Krylov's $L_p$-theory for linear SPDE.

We will paraphrase the condition (BC) to find some reasonable assumptions for $\sigma_1, \sigma_2, b$ to make our regularity argument easy. Note that these functions are scalar functions since we are dealing with the situation $d = 1$. Therefore, we have $L = \frac{1}{2}a\partial_{xx} + b\partial_x$ and $L^* = \frac{1}{2}a\partial_{xx} + (a' - b)\partial_x + (\frac{1}{2}a'' - b')$.

We start by defining some basic spaces. We denote

$$[f]_0 = \sup_{x \in \mathbb{R}} |f(x)|, \quad [f]_\gamma = \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x-y|^\gamma}$$

for $\gamma \in (0, 1]$. Using this notation, we define

$$\|f\|_{C^{0,\gamma}} = [f]_0 + [f]_\gamma, \quad \|f\|_{C^{1,\gamma}} = [f]_0 + [f']_0 + [f']_\gamma$$
$$\|f\|_{C^1} = [f]_0 + [f']_0, \quad \|f\|_{C^2} = [f]_0 + [f']_0 + [f'']_0$$

assuming that $f'$ or $f''$ exist if they appear in the corresponding definition. Then we define the Banach spaces :

$$C^{0,\gamma} = \{f \ : \ \|f\|_{C^{0,\gamma}} < \infty\}, \quad C^{1,\gamma} = \{f \ : \ \|f\|_{C^{1,\gamma}} < \infty\}$$
$$C^1 = \{f \ : \ \|f\|_{C^1} < \infty\}, \quad C^2 = \{f \ : \ \|f\|_{C^2} < \infty\}.$$

**Remark 4.1** *Zygmund spaces $C^{0,\gamma}, C^{1,\gamma}$ are the usual Hölder spaces if $\gamma \in (0, 1)$. It is easy to see that we have $\|f\|_{C^{0,\gamma}} \leq 2\|f\|_{C^{0,1}}, \|f\|_{C^{1,\gamma}} \leq 2\|f\|_{C^{1,1}}$ and $\|f\|_{C^{0,1}} \leq \|f\|_{C^1}, \|f\|_{C^{1,1}} \leq \|f\|_{C^2}$ when $f'$ or $f''$ exists.*

Now, we state assumptions on $\sigma_1, \sigma_2, b$. First, our condition (BC) gives us the following assumption :

$$\sigma_1, \sigma_2 \in C^2, \quad b \in C^1 \tag{4.1}$$

which, in particular, implies $a = \sigma_1^2 + \sigma_2^2 \in C^2$. We also assume that

$$\delta \leq \frac{1}{2}a, \frac{1}{2}\sigma_2^2 \leq K, \quad \|\sigma_1\|_{C^2}, \|\sigma_2\|_{C^2}, \|b\|_{C^1} \leq K \tag{4.2}$$

for some positive constants $\delta, K$.

Next, we recall the basic definitions of some function spaces defined in [7]. In addition to the definition about space of Bessel potentials in the Theorem 1.1, we also

define the following : for $n \in \mathbb{R}$ and $p \in [2, \infty)$ let $H_p^n(l_2)$ be the space with norm

$$\|g\|_{n,p} = \left\| |(I - \Delta)^{n/2} g|_{l_2} \right\|_p$$

for $l_2$-valued functions $g = \{g^k\}$. Then we define

$$\mathbb{H}_p^n(T) = L_p(\Omega \times [0,T], \mathcal{P}, H_p^n) \quad \mathbb{H}_p^n(T, l_2) = L_p(\Omega \times [0,T], \mathcal{P}, H_p^n(l_2))$$

where $\mathcal{P}$ is the predictable $\sigma$-field. We denote $\mathbb{L}_p(T) = \mathbb{H}_p^0(T)$. Let $\{w_t^k : k = 1, 2, \ldots\}$ be a family of independent one-dimensional Brownian motions.

We say $u \in \mathcal{H}_p^n(T)$ if $\partial_{xx} u \in \mathbb{H}_p^{n-2}(T)$ and $u(0, \cdot) \in L_p(\Omega, H_p^{n-2/p})$ and there exists $(f, g) \in \mathbb{H}_p^{n-2}(T) \times \mathbb{H}_p^{n-1}(T, l_2)$ such that $\forall \phi \in C_0^\infty(\mathbb{R})$, (a.s.)

$$\langle u_t, \phi \rangle = \langle u_0, \phi \rangle + \int_0^t \langle f_s, \phi \rangle \, ds + \sum_{k=0}^{\infty} \int_0^t \langle g_s^k, \phi \rangle \, dw_s^k$$

holds for all $t \leq T$. We denote

$$\|u\|_{\mathcal{H}_p^n(T)} = \|\partial_{xx} u\|_{\mathbb{H}_p^{n-2}(T)} + \|f\|_{\mathbb{H}_p^{n-2}(T)} + \|g\|_{\mathbb{H}_p^{n-1}(T, l_2)} + \left( \mathbb{E} \, \|u_0\|_{n-2/p,p}^p \right)^{1/p}$$

Reader can find motivation of this definition and detailed remarks in [7].

Now, we fix $\epsilon \in (0, \frac{1}{4})$ and proceed to the *Proof of Theorem 1.1 (iii)* : First, we freeze the nonlinear term of SPDE (1.6) and consider the following auxiliary linear SPDE for $Y_t(x)$:

$$\begin{cases} \partial_t Y = L^* Y + \sqrt{X} \dot{B}_{tx} \\ Y_0 = \mu \end{cases} \tag{4.3}$$

where we assume $\mu \in H_p^{\frac{1}{2} - \epsilon - 2/p}$.

Then $Z = X - Y$ satisfies

$$\begin{cases} \partial_t Z = L^* Z - (\partial_x(\sigma_1 Z) + \partial_x(\sigma_1 Y)) \, \dot{W}_t \\ Z_0 = 0. \end{cases} \tag{4.4}$$

We apply Theorem 8.5 of [7] to (4.3). To do this we need the coefficients of $L^*$ and $\sqrt{X}$ to satisfy

$$\|a\|_{C^{1,1}} < \infty, \quad \|a' - b\|_{C^{0,1}} < \infty, \quad [\frac{1}{2} a'' - b']_0 < \infty, \quad \|\sqrt{X}\|_{\mathbb{L}_p(T)} < \infty.$$

In fact, we have

$$\|a\|_{C^{1,1}} \leq K, \quad \|a' - b\|_{C^{0,1}} \leq 2K, \quad [\frac{1}{2}a'' - b']_0 \leq 2K$$

by our assumptions (4.1) and (4.2) and Remark 4.1. We will prove $\|\sqrt{X}\|_{\mathbb{L}_p(T)} < \infty$ later and take this for granted in this proof.

Now, by Theorem 8.5 of [7] to (4.3) and the first assertion of Lemma 8.4 and the fact that $\mu$ is nonrandom, we have a unique solution $Y$ in $\mathcal{H}_p^{\frac{1}{2}-\epsilon}(T)$ with estimate

$$\|Y\|_{\mathcal{H}_p^{\frac{1}{2}-\epsilon}(T)} \leq N(\|\sqrt{X}\|_{\mathbb{L}_p(T)} + \|\mu\|_{\frac{1}{2}-\epsilon-2/p,p}) \tag{4.5}$$

where $N$ depends only on $\epsilon, p, \delta, K, T$.

Now we use Theorem 5.1 in [7] for equation (4.4) above with $n = -\frac{3}{2} - \epsilon \in (-2, -\frac{3}{2})$. Note $\partial_x(\sigma_1 Z) = \sigma_1 \partial_x Z + \partial_x \sigma_1 Z$ . If we read [7] carefully, we can see that the following conditions are required :

(i)

$$\delta \leq \frac{1}{2}a - \frac{1}{2}\sigma_1^2 (= \frac{1}{2}\sigma_2^2) \leq K_1$$

for some positive $\delta, K_1$.

(ii) $a, \sigma_1$ are Lipschitz continuous with Lipschitz constant $K_1$.

(iii) $a \in C^{1,\gamma_1}, \sigma_1 \in C^{0,\gamma_2}$ for some $\gamma_1, \gamma_2 \in (0,1)$ and $\|a\|_{C^{1,\gamma_1}} + \|\sigma\|_{C^{0,\gamma_2}} \leq K_1$

(iv) $\|a' - b\|_{C^{0,\gamma_3}} + [\frac{1}{2}(a'' - b')]_0 + [\partial_x \sigma_1]_0 \leq K_1$ for some $\gamma_3 \in (0,1)$.

(v) $\partial_x(\sigma_1 Y) \in \mathbb{H}_p^{n+1}(T)$ $(= \mathbb{H}_p^{-\frac{1}{2}-\epsilon}(T))$.

But, conditions (i) through (iv) are satisfied under (4.1) and (4.2) and Remark 4.1. Note that we can take some constant multiple of $K^2$ as $K_1$. On the other hand, (v) is

also satisfied. For

$$
\begin{aligned}
\|\partial_x(\sigma_1 Y)\|_{\mathbb{H}_p^{-\frac{1}{2}-\epsilon}(T)} &\leq N\|\sigma_1 Y\|_{\mathbb{H}_p^{\frac{1}{2}-\epsilon}(T)} & (4.6)\\
&\leq N\|\sigma\|_{C^{0,\frac{1}{2}-\epsilon+\frac{1}{4}}}\|Y\|_{\mathbb{H}_p^{\frac{1}{2}-\epsilon}(T)} & (4.7)\\
&\leq N\|\sigma\|_{C^1}\|Y\|_{\mathbb{H}_p^{\frac{1}{2}-\epsilon}(T)} & (4.8)\\
&\leq N\|Y\|_{\mathcal{H}_p^{\frac{1}{2}-\epsilon}(T)} & (4.9)\\
&\leq N\|\sqrt{X}\|_{\mathbb{L}_p(T)} + N\|\mu\|_{\frac{1}{2}-\epsilon-2/p,p} < \infty. & (4.10)
\end{aligned}
$$

(4.6) follows the observation $\partial_x = \partial_x(I-\Delta)^{-1/2}(I-\Delta)^{1/2}$ and the boundness of the operator $\partial_x(I-\Delta)^{-1/2}$. (4.7) follows Lemma 5.1 (i) in [7]. Up to this step, $N$ only depends on $\epsilon, p$. Note that $\frac{1}{2}-\epsilon+\frac{1}{4}$ is still in $(0,1)$ since $\frac{1}{2}-\epsilon \in (\frac{1}{4},\frac{1}{2})$. Hence, we have (4.8) by (4.2) and Remark 4.1. (4.9) follows Theorem 3.7 in [7] and $N$ depends only on $\epsilon, p, K, T$ now. Finally, (4.5) gives us (4.10) with $N = N(\epsilon, p, \delta, K, T)$.

Therefore, we have a unique solution $Z$ in $\mathcal{H}_p^{\frac{1}{2}-\epsilon}(T)$ with

$$
\|Z\|_{\mathcal{H}_p^{\frac{1}{2}-\epsilon}(T)} \leq N\,\|\partial_x(\sigma_1 Y)\|_{\mathbb{H}_p^{-\frac{1}{2}-\epsilon}(T)} \leq N\|\sqrt{X}\|_{\mathbb{L}_p(T)} + N\|\mu\|_{\frac{1}{2}-\epsilon-2/p,p} \tag{4.11}
$$

where $N = N(\epsilon, p, \delta, K, T)$.

Thus, combining (4.5) and (4.11), we have $X = Y + Z \in \mathcal{H}_p^{\frac{1}{2}-\epsilon}(T)$ with estimate

$$
\|X\|_{\mathcal{H}_p^{\frac{1}{2}-\epsilon}(T)} \leq N\|\sqrt{X}\|_{\mathbb{L}_p(T)} + N\|\mu\|_{\frac{1}{2}-\epsilon-2/p,p}. \tag{4.12}
$$

By the embedding Theorem 7.1 in [7], this implies

$$
\left(E\int_0^T \|X_t\|^p_{C^{\frac{1}{2}-\epsilon-\frac{1}{p}}}dt\right)^{1/p} \leq N\|X\|_{\mathcal{H}_p^{\frac{1}{2}-\epsilon}(T)} \leq N\|\sqrt{X}\|_{\mathbb{L}_p(T)} + N\|\mu\|_{\frac{1}{2}-\epsilon-2/p,p}.
$$

So, for large $p > \frac{1}{\epsilon}$ , we have

$$
\|X_t\|_{C^{\frac{1}{2}-2\epsilon}} < \infty
$$

for (a.e.) $t \in [0,T]$ a.s.. we are done with the proof. ∎

Finally, we use the moment dual to prove that

$$\mathbb{E} \int_0^T \int_{\mathbb{R}} X(t,x)^n dxdt < \infty \tag{4.13}$$

for all $n \in \mathbb{N}$.

Let $n_t$ be a pure-death Markov chain with $n_0 = 0$ and, at a rate $\frac{1}{2}n(n-1)$, jumps from $n$ to $n-1$. Let $0 = \tau_0 < \tau_1 < \cdots < \tau_{n-1}$ be the jump times. Let $f_0 = \delta_x^{\otimes n}$ and for $t < \tau_1$, $f_t(y) = p_0^n(t, (x, \cdots, x), y)$, $\forall y \in \mathbb{R}^n$ where $p_0^n$ is the transition function of the $n$-dimensional diffusion $(\eta_1(t), \cdots, \eta_n(t))$. For $f \in C(\mathbb{R}^n)$, let $G_{ij} f \in C(\mathbb{R}^{n-1})$ be given by

$$G_{ij} f(y_1, \cdots, y_{n-2}, y_{n-1}) = f(y_1, \cdots, y_{n-1}, \cdots, y_{n-1}, \cdots, y_{n-2})$$

where $y_{n-1}$ is at $i$th and $j$th position. Let

$$f_{\tau_1} = \Gamma_1 f_{\tau_1 -}$$

where $\Gamma_1$ is a random element taking values in $\{G_{ij} : 1 \leq i < j \leq n\}$ uniformly. We continue this procedure to get the process $f_t$. Replace $f_0$ by a smooth function $f_0^k \geq 0$ approximating $f_0$. Denote the process constructed above with $f_0^k$ in place of $f_0$ by $f_t^k$. Similar to Theorem 11 in Xiong and Zhou [15], we have

$$\mathbb{E} \left\langle X_t^{\otimes n}, f_0^k \right\rangle = \mathbb{E} \left( \left\langle \mu^{\otimes n_t}, f_t^k \right\rangle \exp\left( \frac{1}{2} \int_0^t n_s(n_s - 1) ds \right) \right).$$

Taking limits and using Fatou's lemma, we have

$$\begin{aligned} \mathbb{E}\, X(t,x)^n &\leq \mathbb{E} \left( \left\langle \mu^{\otimes n_t}, f_t \right\rangle \exp\left( \frac{1}{2} \int_0^t n_s(n_s - 1) ds \right) \right) \\ &\leq \exp\left( \frac{1}{2} n(n-1)t \right) \sum_{i=1}^n \mathbb{E} \left( \left\langle \mu^{\otimes n_t}, f_t \right\rangle 1_{\tau_{i-1} \leq t < \tau_i} \right). \end{aligned}$$

Let $i = 3$. Then

$$f_t(x_1, \cdots, x_{n-2}) \leq c \int_{\mathbb{R}^{n-2}} \Pi_{i=1}^{n-2} \varphi_{t-\tau_2}(x_i - y_i) \Gamma_2 f_{\tau_2 -}(y) dy$$

$$
\begin{aligned}
&\le c\int_{\mathbb{R}^{n-2}} \Pi_{i=1}^{n-2}\varphi_{t-\tau_2}(x_i-y_i)\sum_{1\le k<\ell}\frac{2}{(n-2)(n-3)} \\
&\qquad f_{\tau_2-}(y_1,\cdots,y_{n-2},\cdots,y_{n-2},\cdots,y_{n-3})dy \\
&\le c\int_{\mathbb{R}^{n-2}} \Pi_{i=1}^{n-2}\varphi_{t-\tau_2}(x_i-y_i)\sum_{1\le k<\ell}\frac{2}{(n-2)(n-3)} \\
&\qquad \int_{\mathbb{R}^{n-1}} \Pi_{j=1}^{n-1}\varphi_{\tau_2-\tau_1}(y_j-z_j)\varphi_{\tau_1}(z_1-x)\cdots \\
&\qquad \cdots\varphi_{\tau_1}(z_{n-2}-x)\varphi_{\tau_1}(z_{n-1}-x)^2dz.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\langle\mu^{\otimes n-2},f_t\rangle &\le c\int_{\mathbb{R}}\varphi_{t-\tau_2}(x_{n-2}-y_{n-2})\mu(dx_{n-2})\int_{\mathbb{R}}dy_{n-2}\sum_{1\le k<\ell}\frac{2}{(n-2)(n-3)} \\
&\qquad \varphi_{\tau_2-\tau_1}(y_{n-2}-z_k)\varphi_{\tau_2-\tau_1}(y_{n-2}-z_\ell) \\
&\qquad \int_{\mathbb{R}^{n-1}}\varphi_{\tau_1}(z_1-x)\cdots\varphi_{\tau_1}(z_{n-2}-x)\varphi_{\tau_1}(z_{n-1}-x)^2 \\
&\le c\int_{\mathbb{R}}\varphi_{t-\tau_2}(x_{n-2}-y_{n-2})\mu(dx_{n-2})\int_{\mathbb{R}}dy_{n-2}\frac{1}{\sqrt{\tau_1(\tau_2-\tau_1)}}\varphi_{\tau_2}(y_{n-2}-x).
\end{aligned}
$$

Therefore

$$
\int_{\mathbb{R}}\mathbb{E}\ \langle\mu^{\otimes n_t},f_t\rangle\,1_{\tau_2\le t<\tau_3}dx\le c\mu(\mathbb{R})\mathbb{E}\ \frac{1}{\sqrt{\tau_1(\tau_2-\tau_1)}}<\infty.
$$

The other terms can be proved similarly. This finishes the proof of Theorem 1.1.

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